國立中央大學九十二學年度碩士班考試入學招生試題卷 共 1 頁 第 1 頁

系所別: 數學系 科目: 高等微積分

In these problems, $\mathbb{R}$ denotes the set of all real numbers.

1. (15%) Let $f_n(x) = \sin x^n$, $n = 1, 2, \cdots$, for $x \in (0,1)$.

(a) Find the function $f_0(x)$ such that $f_0(x) = \lim_{n\to\infty} f_n(x)$ for every $x \in (0,1)$.

(b) Does $f_n(x)$ converge uniformly to $f_0(x)$ on $(0,1)$? Give sufficient reason to support your answer.

2. (15%) Let V be an open set in $\mathbb{R}^n$, $x_0 \in V$, and $f : V \to \mathbb{R}^m$.

(a) Give the definition that f be differtiable at x_0

(b) Let

$$f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Find $f_x(0,0)$, $f_y(0,0)$ if they exist, and prove that f is not differentiable at $(0,0)$.

參考用

3. (15%) Let E be the subset of the Euclidean space $\mathbb{R}^2$ defined by

$$E = \{(x,y) : y = \sin \tfrac{1}{x},\ x \in (0,1]\}.$$

(a) Find the closure $\overline{E}$ of E, and give your reason.

(b) Which one of the sets E and $\overline{E}$ is connected? Why?

(c) Which one of the sets E and $\overline{E}$ is compact? Why?

4. (20%) Let $\mathbb{R}^n$, $\mathbb{R}^m$ be Euclidean spaces and $D \subset \mathbb{R}^n$; $f : D \to \mathbb{R}^m$ be uniformly continuous on D.

(a) Prove that if $\{x_k\}_{k=1}^{\infty}$ is a Cauchy sequence in D, then $\{f(x_k)\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}^m$.

(b) If D is dense in $\mathbb{R}^n$, then show that f has a continuous extension to $\mathbb{R}^n$. That is, there is a continuous function $g : \mathbb{R}^n \to \mathbb{R}^m$ such that $g(x) = f(x)$ for all $x \in D$.

5. (20%) (a) Prove that the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} \sin \frac{x}{k+1}$ converges on $\mathbb{R}$.

(b) If

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k} \sin \frac{x}{k+1}, \qquad x \in \mathbb{R},$$

then prove that $|f(x)| \le |x|$.

(c) Prove that the series in (a) converges uniformly on any bounded subset E of $\mathbb{R}$.

6. (15%) Evaluate the surface integral

$$\iint_S x^2\, dy\, dz + y^2\, dz\, dx + z^2\, dx\, dy$$

where S is the surface, i.e., the set of all boundary points, of the solid

$$V = \{(x,y,z) : 0 \le x^2 + y^2 \le 1, \text{ and } 0 \le z \le 1\}.$$