maximal degree among common divisors of $f_1(x), f_2(x), \ldots, f_n(x)$. Denote the G.C.D. by $\gcd(f_1(x), f_2(x), \ldots, f_n(x))$. Prove

$$\gcd(f_1(x), f_2(x), \ldots, f_n(x)) = \sum_{i=1}^{n} a_i(x) f_i(x) \text{ for some } a_i(x) \in F[x],\ i = 1, \ldots, n$$

by completing the following steps.

(a) (12 %) Show that $F[x]$ is a principal ideal domain.

(b) (8 %) Let $\mathcal{A}$ be the ideal generated by $f_1(x), f_2(x), \ldots, f_n(x)$. Show that $\mathcal{A}$ is also generated by $\gcd(f_1(x), f_2(x), \ldots, f_n(x))$ and conclude that

$$\gcd(f_1(x), f_2(x), \ldots, f_n(x)) = \sum_{i=1}^{n} a_i(x) f_i(x).$$

6. (10 %) Let a, b be relatively prime non-zero integers such that at least one of a, b is not ± 1. Assume that both a and b are square free integers. Is it true that the polynomial $ax^m - b$ is irreducible in $\mathbb{Q}[x]$ for every positive integer m? Explain your answer.
Note. An integer n is called square free if it has that property that for prime p with $p \mid n$ then $p^2 \nmid n$.

7. (12 %) Let $P(x)$ be an irreducible polynomial in $\mathbb{Q}[x]$. Let $r \in \mathbb{C}$ be a root of $P(x)$. Given

$$\alpha = \frac{a_n r^n + \cdots + a_1 r + a_0}{b_m r^m + \cdots + b_1 r + b_0} \text{ where } a_i, b_j \in \mathbb{Q} \text{ and } b_m r^m + \cdots + b_1 r + b_0 \neq 0.$$

Prove or disprove that there exists a polynomial $f_\alpha(x) \in \mathbb{Q}[x]$ such that $\alpha = f_\alpha(r)$.