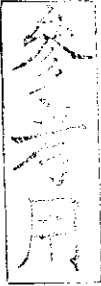科目：高等微積分(1001)　　校系所組： 中大數學系甲組　清大數學系純粹數學組
清大數學系應用數學組

Advanced Calculus

本試卷共八題

1. (14 points) Let $E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

   (a) (9 points) Show that $E(x)$ is differentiable on $\mathbb{R}$.

   (b) (5 points) Prove that the function $y = E(x)$ satisfies the initial value problem
   $$y' - y = 0, \qquad y(0) = 1.$$

2. (12 points) Prove that the function $\frac{\sin x}{x}$ is improperly integrable on $[1, \infty)$.

3. (14 points) Prove that
   $$f(x,y) = \begin{cases} \frac{x^3 - xy^2}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$
   is continuous, has first-order partial derivatives everywhere on $\mathbb{R}^2$, but f is not differentiable at $(0,0)$.

4. (12 points) Let $f(x,y) = (e^x + e^y, e^x - e^y)$. Show that f is locally invertible around every point of $\mathbb{R}^2$. Does f have a global inverse on $\mathbb{R}^2$ itself? Give your reason.

5. (12 points) Let
   $$F(T) = \frac{1}{2T} \int_{-T}^{T} e^{-x^2} dx, \; T > 0.$$
   Show that $F(T)$ is a strictly decreasing function in T.

6. (12 points) Find the limit
   $$\lim_{n \to \infty} \frac{\pi}{n^4} \sum_{1 \leq k, l \leq n} k^2 \cos(\frac{l}{n}\pi).$$
   Explain each step of your computation.

7. (12 points) Let A be a $n \times n$ real matrix and $\det A < 0$. Show that there is a positive number δ such that
   $$\|Ax\| \geq \delta \|\mathbf{x}\|, \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

8. (12 points) Evaluate the line integral
   $$I = \oint_C \frac{(2x + y)dx + (2y - x)dy}{x^2 + y^2},$$
   where C is the oriented counterclockwise circle $\{(x,y) : x^2 + y^2 = 1\}$.