科目：數值分析(1005)　　校系所組：中大數學系乙組

參考用

Instructions: Do all 4 problems. Show your work.

1. **(Numerical linear algebra)**

Consider the linear system, $Ax = b$, where

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

and

$$b = [1, -1, 3]^T$$

(a) Use Gaussian elimination to obtain the decomposition

$$A = LDL^T,$$

where L is a unit lower triangular matrix and D is a diagonal matrix. Show all intermediate steps as well as pivot elements and multipliers. (10 pts)

(b) Find the first two iterations of the **Jacobi** and the **Gauss-Seidel methods** for the above linear system, using the initial vector, $[x_1^{(0)}, x_2^{(0)}, x_3^{(0)}]^T = [0, 0, 0]^T$. (10 pts)

(c) Prove the Jacobi method applied to the above system will converge for any initial vector, $[x_1^{(0)}, x_2^{(0)}, x_3^{(0)}]^T$. If you use any theorem in the proof, please state that theorem completely. (10 pts)

2. **(Numerical Differential Equations)**

(a) Assume that $u \in C^4[x_0 - h, x_0 + h]$. Use Taylor's theorem to derive the second-order central difference approximation for the second derivative of $u(x)$ at $x = x_0$, including the error term, for $h > 0$:

$$u''(x_0) = \frac{u(x_0 + h) - 2u(x_0) + u(x_0 - h)}{h^2} - \frac{h^2}{12}u^{(4)}(\xi), \qquad (1)$$

where $\xi \in (x_0 - h, x_0 - h)$. (10 pts)

(b) Solve the following two-point boundary value problem:

$$\begin{cases} u'' + 4x = 0 \\ u(0) = 0 \qquad u(1) = 1 \end{cases}$$

for $u(1/2)$ approximately using Formula (1) with $h = 1/2$. (10 pts)

注意：背面有試題

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3. (**Numerical Integration**)

(a) Verify that the polynomials

$$\begin{aligned} p_0(x) &= 1 \\ p_1(x) &= x \\ p_2(x) &= x^2 - 1/3 \end{aligned}$$

are orthogonal with respect to the weight function $w(x) = 1$ on the interval $[-1, 1]$. (10 pts)

(b) Find the Gaussian quadrature formula for $n = 2$ on the interval $[-1, 1]$ with the weight function $w(x) = 1$ (10 pts). That is, find the weights c_i and the nodes x_i for the formula

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{2} c_i f(x_i)$$

What is the degree of precision of this formula? (5 pts). Write down the corresponding rule for an arbitrary interval $[a, b]$ (5 pts).

4. (**Interpolation**)

Consider the table

x	1	3	2	4
y	−2	−3	−1	2

(a) Find the Lagrange form of the interpolation polynomial of degree 3 passing the points given in the above table. (10 pts)

(b) Redo part (a) by using the Newton form of the interpolation polynomial. (10 pts)