台灣聯合大學系統93學年度學士班轉學生考試命題紙

科目：高等微積分　類組別：C-2　共 1 頁第 1 頁　*請在試卷答案卷（卡）內作答

(1) Let $\{a_n\}$ be a sequence of positive real numbers. Prove that

$$\limsup_{n\to\infty} \sqrt[n]{a_n} \le \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}. \quad (10\%)$$

(2) Let

$$\begin{aligned}
A &= \{(x,0)\in\mathbb{R}^2 \;:\; 0<x<1\},\\
B &= \{(x,y)\in\mathbb{R}^2 \;:\; 0<xy<1\},\\
C &= \{(x,y)\in\mathbb{R}^2 \;:\; x \text{ is rational and } 0\le y\le 1\},\\
D &= \{(0,0)\}\cup\{(\tfrac{1}{x},0)\in\mathbb{R}^2 \;:\; x=1,2,3,\cdots\},\\
E &= \{(3x+2y, 8x-9y)\in\mathbb{R}^2 \;:\; (x,y)\in\mathbb{R}^2, x^2+y^2<1\},\\
F &= \{(e^{\sin(xy)}, e^{\cos(x+y)})\in\mathbb{R}^2 \;:\; (x,y)\in\mathbb{R}^2, x^2+y^2\le 2\}.
\end{aligned}$$

(a) Which sets are open in $\mathbb{R}^2$? (Don't need to prove.) (4%)
(b) Which sets are compact in $\mathbb{R}^2$? (Don't need to prove.) (4%)
(c) Which sets are connected in $\mathbb{R}^2$? (Don't need to prove.) (4%)
(d) Which sets are complete in $\mathbb{R}^2$? (Don't need to prove.) (4%)

(3) Is the function $f(x)=\sqrt[3]{x^2}$ uniformly continuous on $\mathbb{R}$? Give your proof. (15%)

(4) Determine whether the function $f(x)=\sum_{n=0}^{\infty}\frac{x^{3n}\sin nx}{(n!)^3}$ is continuous on $\mathbb{R}$. Give your proof. (10%)

(5) Let $f:(-1,\infty)\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases}\dfrac{|x|}{4} & \text{if } x \text{ is rational and } -1<x<\infty;\\[2mm] \dfrac{3x^2}{9+4x^2} & \text{if } x \text{ is irrational and } -1<x<\infty.\end{cases}$$

Prove that there exists one and only one point $c\in(-1,\infty)$ such that $f'(c)$ exists, and, find c and $f'(c)$. (15%)

(6) Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by setting $f(0,0)=0$ and

$$f(x,y)=\frac{xy(x^2-y^2)}{x^2+y^2} \quad \text{if} \quad (x,y)\ne(0,0).$$

(a) Do $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(0,0)$? Give your proof. (4%)
(b) Calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(x,y)\ne(0,0)$. (4%)
(c) Is f of class C^1 on $\mathbb{R}^2$? Give your proof. (10%)
(d) Is f of class C^2 on $\mathbb{R}^2$? Give your proof. (6%)
(Recall that a function is said to be of class C^r if the first r derivatives exist and are continuous.)

(7) Let $f:[0,2]\to\mathbb{R}$ be Riemann integrable on $[0,2]$. Assume that the set $A=\{x\in[0,2]: f(x)=2\}$ is dense in $[0,2]$. Find the integral $\int_0^2 f(x)dx$. (10%)