Number theory

1. $k_1, k_2, r, s \in N$, $k_1 \le 2^{k_2}$.

   Suppose that $k_1 | r^s$. Show that $k_1 | r^{k_2}$

   $Proof_1$:

   We have $\begin{cases} k_1 \le 2^{k_2}, \\ k_1 | r^s \cdots\cdots (1) \end{cases}$

   Suppose, on the contrary, that $\quad k_1 \nmid r^{k_2} \cdots\cdots$ (2)

   We will show that (2) leads to a contradiction.

   (2) $\Rightarrow$ there exist a prime p and $s \in N$

   such that $p^s | k_1 \cdots\cdots$ (3)

   $p^s \nmid r^{k_2} \cdots\cdots$ (4)

   (3) $\Rightarrow p^s | k_1$

   $\Rightarrow p^s | r^s \quad (\because (1))$

   $\Rightarrow p | r^s$

   $\Rightarrow p | r \quad (\because$ p is prime)

   $\Rightarrow p^s | r^s$

   $\Rightarrow k_2 < s \quad (\because (4))$

   $\Rightarrow p^{k_2} < p^s$

   $\Rightarrow 2^{k_2} \le p^{k_2} < p^s \le k_1 \quad (\because (4))$

   $\Rightarrow 2^{k_2} < k_1$

   $\rightarrow\leftarrow k_1 \le 2^{k_2}$. $\square$

   $Proof_2$:

   $k_1 \le 2^{k_2}$

   $\Rightarrow$ any prime factor of k_1 has exponent $\le k_2 \quad \cdots\cdots$ (1)

   ($\because$ Let $p_1 | k_1$, p_1 is prime and $k_1 = p_1^{e_1} p_2^{e_2} \cdots \cdot p_s^{e_s}$

   where $p_1, \cdots, p_s$ are distinct primes and $e_1, \cdots, e_s \in \mathbb{N}$

   Then $\quad p_1^{e_1} \le 2^{k_2} \;\Rightarrow\; e_1 \le k_2 \quad (\because p_1 \ge 2)$)

   $k_1 | r^s$

   $\Rightarrow$ any prime factor of k_1 is a prime factor of $r \quad \cdots\cdots$ (2)

   (1), (2) $\Rightarrow$ the result. $\square$

2. (1) Show that $9 \cdot 2^n \cdot 10^{n-1} + 1$ is divisible by 19 for all positive integers n .

   (2) Show that the (n + 1)-digit

   $a_n a_{n-1} \cdots a_1 a_0$ ($= a_n 10^n + \cdots + a_1 10 + a_0$) is divisible by 19 if and only if $10a_n + a_{n-1} + 2a_{n-2} + 4a_{n-3} + \cdots + 2^{n-1}a_0$ is.

Proof :

(1) Prove by induction on n .

The case $n = 1$ is trivial.

Suppose $9 \cdot 2^n \cdot 10^{n-1} + 1$ is divisible by 19 .

Then
$$\begin{aligned} 9 \cdot 2^{n+1} \cdot 10^n + 1 &= 2 \cdot 10 \cdot 9 \cdot 2^n \cdot 10^{n-1} + 1 \\ &= 20\,(9 \cdot 2^n \cdot 10^{n-1} + 1) - 19 \end{aligned}$$
is divisible by 19 . □

(2) Note that

$$a_n 10^n + \cdots + a_1 10 + a_0 + 9 \cdot 10^{n-2}(10a_n + a_{n-1} + 2a_{n-2} + 4a_{n-3} + \cdots + 2^{n-1}a_0)$$
$$= 19 \cdot 10^{n-1}\, a_n + 19 \cdot 10^{n-2}\, a_{n-1} + \sum_{m=1}^{n-1} 10^{n-1-m}\, a_{n-1-m} + \sum_{m=1}^{n-1} 9 \cdot 10^{n-2} \cdot 2^m\, a_{n-1-m}$$
$$= 19 \cdot 10^{n-1}\, a_n + 19 \cdot 10^{n-2}\, a_{n-1} + \sum_{m=1}^{n-1} 10^{n-1-m}\,(1 + 9 \cdot 10^{m-1} \cdot 2^m)\, a_{n-1-m}$$

is divisible by 19 ($\because$ by (1), each $1 + 9 \cdot 10^{m-1} \cdot 2^m$ is divisible by 19)

Thus $a_n 10^n + \cdots + a_1 10 + a_0$ is divisible by 19

$\Leftrightarrow$ $9 \cdot 10^{n-2}(10a_n + a_{n-1} + 2a_{n-2} + 4a_{n-3} + \cdots + 2^{n-1}a_0)$ is divisible by 19

$\Leftrightarrow$ $10a_n + a_{n-1} + 2a_{n-2} + 4a_{n-3} + \cdots + 2^{n-1}a_0$ is divisible by 19

($\because$ $9 \cdot 10^{n-2}$ and 19 are relatively prime) □

3. Let e, l, n be positive integers such that $2e|(e+1)ln$ and $2e|ln^2$. Show that $2e|ln$.

Proof :

Case 1. e is even.

Since $gcd(e, e+1) = 1$ and $e+1$ is odd, we have $gcd(2e, e+1) = 1$. It follows from $2e|(e+1)ln$ that $2e|ln$.

Case 2. e is odd.

From $2e|(e+1)ln$, we have $e|ln$. From $2e|ln^2$, we have $2|ln^2$; hence $2|ln$. Combining $e|ln$ and $2|ln$, we have $2e|ln$ since e is odd. □