Combinatorics

1. $a_1 > a_2 > a_3 > \cdots > a_7$ are positive integers such that
   $a_1 + a_2 + a_3 + \cdots + a_7 = 100.$
   Show that $a_1 + a_2 + a_3 \geq 50$
   Proof :
   Consider two cases: Case1: $a_3 \geq 16$, Case2: $a_3 \leq 15$.
   Case 1: $a_3 \geq 16$
   Then $a_1 + a_2 + a_3 \geq 18 + 17 + 16 = 51$
   Case 2: $a_3 \leq 15$
   Then $a_4 + a_5 + a_6 + a_7 \leq 14 + 13 + 12 + 11 = 50$
   $$\begin{aligned} \Rightarrow a_1 + a_2 + a_3 &= (a_1 + \cdots a_7) - (a_4 + \cdots a_7) \\ &\geq 100 - 50 \\ &= 50 \end{aligned}$$ □

2. $a_1 < a_2 < a_3 < \cdots < a_7 \leq 1706$ are positive integers.
   Show that $a_1 + a_{i+1} < 4a_i$ for some $i = 2, 3 \cdots, 6$.
   Proof :
   Suppose, on the contrary, that
   $a_1 + a_{i+1} \geq 4a_i$ for all $i = 2, 3 \cdots, 6$.
   Then $a_{i+1} \geq 4a_i - a_1$ for $i = 2, 3 \cdots, 6$.
   $$\begin{aligned} \therefore a_3 &\geq 4a_2 - a_1 \geq 4(a_1 + 1) - a_1 = 3a_1 + 4 , \\ a_4 &\geq 4a_3 - a_1 \geq 4(3a_1 + 4) - a_1 = 11a_1 + 16 , \\ a_5 &\geq 4a_4 - a_1 \geq 4(11a_1 + 16) - a_1 = 43a_1 + 64 , \\ a_6 &\geq 4a_5 - a_1 \geq 4(43a_1 + 64) - a_1 = 171a_1 + 256 , \\ a_7 &\geq 4a_6 - a_1 \geq 4(171a_1 + 256) - a_1 \\ &= 683a_1 + 1024 \geq 1707 \text{ , a contradiction.} \end{aligned}$$ □

3. (1) Given 69 distinct positive integers not exceeding 100 ,
   prove that one can choose four of them a,b,c,d such that a+b+c=d .
   (2) Show that (1) is not true if 69 is replaced by 68.
   Proof :
   (1) Let $a_1 < a_2 < a_3 < \cdots < a_{69} \leq 100$ be positive integers $\cdots$
   We will show that $a_1 + a_2 + a_i = a_j$ for some $2 < i < j$
   $\Rightarrow a_1 \leq 32$

$$\therefore \begin{cases} 1 \le a_3 + a_1 < a_4 + a_1 \cdots < a_{69} + a_1 \le 100 + 32 = 132 \\ 1 \le a_3 - a_2 < a_4 - a_2 \cdots < a_{69} - a_2 \le 100 \le 132 \end{cases}$$

$\Rightarrow a_i + a_1 = a_j - a_2$ for some $3 \le i$, some j (why?)

$\therefore a_1 + a_2 + a_i = a_j$ for some $3 \le i < j$

(2) Consider the numbers 33, 34, 35 $\cdots$, 100. □

4. Let $a_1 < a_2 < \cdots < a_n$ $(n \ge 2)$ be positive integers.
Show that there exist $1 \le i < j \le n$ such that
$a_i + a_j$, $a_j - a_i \notin \{a_1, \cdots\cdots, a_n\} - \{a_i, a_j\}$.

Proof :

Suppose, to the contrary, that
for every $1 \le i < j \le n$,
either $a_i + a_j$ or $a_j - a_i \in \{a_1, \cdots\cdots, a_n\} - \{a_i, a_j\}$ $\cdots\cdots\cdots$(1)

For $1 \le i \le n-1$, $a_i + a_n > a_n$ $(\because a_i > 0)$

$\Rightarrow$ For $1 \le i \le n-1$, $a_i + a_n \notin \{a_1, \cdots\cdots, a_n\}$ $(\because a_1 < a_2 < \cdots < a_n)$

$\Rightarrow$ For $1 \le i \le n-1$, $a_i + a_n \notin \{a_1, \cdots\cdots, a_n\} - \{a_i, a_n\}$

$\Rightarrow$ For $1 \le i \le n-1$, $a_n - a_i \in \{a_1, \cdots\cdots, a_n\} - \{a_i, a_n\}$ $(\because (1))$

$\cdots\cdots\cdots$ (2)

$\Rightarrow$ For $1 \le i \le n-1$, $a_n - a_i \in \{a_1, a_2, \cdots\cdots, a_n\} - \{a_n\}$
$= \{a_1, a_2, \cdots\cdots, a_{n-1}\}$

$$\therefore \begin{cases} a_n - a_{n-1}, a_n - a_{n-2}, \cdots, a_n - a_2, a_n - a_1 \in \{a_1, a_2, \cdots\cdots, a_{n-1}\} \\ a_1 < a_2 < \cdots < a_{n-2} < a_{n-1} \ (by\ \ assumption) \\ \Rightarrow a_n - a_{n-1} < a_n - a_{n-2} < \cdots < a_n - a_1 \end{cases}$$

Hence $a_n - a_{n-1} = a_1$, $a_n - a_{n-2} = a_2$, $a_n - a_{n-3} = a_3, \cdots\cdots$,
$a_n - a_1 = a_{n-1}$ $\cdots\cdots\cdots\cdots$(3)

Case 1. n is even (i.e., n = 2s, s $\in$ N)

(3) $\Rightarrow a_n - a_{n-s} = a_s$
$\Rightarrow a_n - a_s = a_s$
$\Rightarrow a_n - a_s \notin \{a_1, \cdots\cdots, a_n\} - \{a_s, a_n\}$
$\rightarrow\leftarrow$ (2)

Case 2. n is odd (i.e., n = 2s +1, s $\in$ N)

(3) $\Rightarrow a_{2s+1} - a_{2s} = a_1$

$\Rightarrow a_{2s} + a_1 = a_{2s+1}$

$\Rightarrow$ For $2 \le k, \quad a_{2s} + a_k > a_{2s+1} \quad (\because a_1 < a_2 < \cdots)$

$\Rightarrow$ For $2 \le k, \quad a_{2s} + a_k \notin \{a_1, \cdots\cdots, a_{2s+1}\} - \{a_{2s}, a_k\}$

$(\because a_1 < \cdots < a_{2s+1})$

$\Rightarrow$ For $2 \le k \le 2s-1, \quad a_{2s} - a_k \in \{a_1, \cdots, a_{2s+1}\} - \{a_{2s}, a_k\} \quad (\because (1))$

$\cdots\cdots\cdots\cdots (4)$

$\Rightarrow a_{2s} - a_2, a_{2s} - a_3, \cdots, a_{2s} - a_{2s-1} \in \{a_1, \cdots\cdots, a_{2s+1}\}$

$\cdots\cdots\cdots\cdots (5)$

(3) $\Rightarrow a_{2s+1} - a_{2s-1} = a_2$

$\Rightarrow a_{2s+1} - a_2 = a_{2s-1}$

$\Rightarrow a_{2s} - a_2 < a_{2s-1} \quad (\because a_{2s} < a_{2s+1})$

$\Rightarrow a_{2s} - a_{2s-1} < \cdots < a_{2s} - a_3 < a_{2s} - a_2 < a_{2s-1} \quad (\because a_2 < a_3 < \cdots)$

$\Rightarrow a_{2s} - a_{2s-1}, \cdots, a_{2s} - a_3, a_{2s} - a_2 \in \{a_1, \cdots\cdots, a_{2s-2}\} \quad (\because (5))$

$$\text{We have } \begin{cases} a_{2s} - a_{2s-1}, \cdots, a_{2s} - a_3, a_{2s} - a_2 \in \{a_1, \cdots\cdots, a_{2s-2}\} \\ a_1 < a_2 < \cdots < a_{2s-2} \\ a_{2s} - a_{2s-1} < a_{2s} - a_{2s-2} < \cdots < a_{2s} - a_2 \end{cases}$$

$\Rightarrow a_{2s} - a_{2s-1} = a_1,$

$a_{2s} - a_{2s-2} = a_2,$

$\vdots$

$a_{2s} - a_s = a_s$

$\Rightarrow a_{2s} - a_s \notin \{a_1, \cdots\cdots, a_n\} - \{a_{2s}, a_s\}$

$\rightarrow\leftarrow (4) \qquad \square$

5. $k, n \in N, \ \ n \ge 2 + k(k-1),$

$A_1, A_2, \cdots, A_n$ are sets such that

$|A_i| = k$ for $i = 1, 2, \cdots, n.$ and $|A_i \cap A_j| = 1$ for $1 \le i < j \le n$

Show that $A_1 \cap A_2 \cap \cdots \cap A_n \ne \emptyset$

Proof :

$|A_1| = k, |A_1 \cap A_2| = |A_1 \cap A_3| = \cdots = \left|A_1 \cap A_{2+k(k-1)}\right| = 1$

$\Rightarrow$ At least one element in A_1 appears in k sets of $A_2, A_3, \cdots, A_{2+k(k-1)}$, (why?)

say $x \in A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_{k+1}$

$\Rightarrow \{x\} = A_i \cap A_j$ for $1 \le i < j \le k+1 \quad (\because |A_i \cap A_j| = 1) \quad \cdots (1)$

Then $x \in A_l$ for $k+2 \le l \le n \quad \cdots\cdots (2)$

($\because$ Suppose, on the contrary, that $x \notin A_l$ for some $k+2 \le l \le n$

Then $A_i \cap A_j \cap A_l = \emptyset$ for $1 \le i < j \le k+1 \quad (\because (1))$

$\Rightarrow A_l \cap A_1, A_l \cap A_2, \cdots, A_l \cap A_{k+1}$ are pairwise disjoint $\cdots (3)$

We have $A_l \supset A_l \cap (A_1 \cup A_2 \cup \cdots \cup A_{k+1})$

$$= (A_l \cap A_1) \cup (A_l \cap A_2) \cup \cdots \cup (A_l \cap A_{k+1})$$

$$\Rightarrow |A_l| \geq |A_l \cap A_1| + |A_l \cap A_2| + \cdots + |A_l \cap A_{k+1}| \quad (\because (3))$$

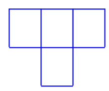
$$= k + 1 \quad (\because |A_l \cap A_i| = 1, \quad \text{for } i = 1, 2, \cdots, k+1)$$

$$\rightarrow\leftarrow |A_l| = k \qquad)$$

(1),(2) $\Rightarrow$ $x \in A_1 \cap A_2 \cap \cdots \cap A_{k+1} \cap \cdots \cap A_n$.

$\therefore A_1 \cap A_2 \cap \cdots \cap A_n \neq \emptyset$ □

6. A T-tetromino is a configuration of four unit squares arranged in the following shape:



Suppose that an $m \times n$ rectangular board can be tiled with T-tetrominos.
Show that $8|mn$.

Proof :

Let a tiling be given.

$m \times n$ rectangular board can be tiled with T-tetrominos.

$\Rightarrow 4|mn \quad (\because$ T-tetrominos has four unit squares)

$\Rightarrow 2|m$ or $2|n \ \cdots \ (1)$

Imagine the board as a red and white checkerboard.

Then

(i) the board has the same number of red squares and white squares. ($\because$ (1)) (why?)

(ii) each T-tetrominos in the tiling either covers 3 red squares and 1 white square
or covers 1 red square and 3 white squares;
in the former case, we say that the T-tetromino is of type 1,
and in the latter case we say that the T-tetromino is of type 2.

(i),(ii) $\Rightarrow$ in the tiling, there are the same number of T-tetrominos of type 1
and those of type 2 (why?)

$\Rightarrow$ in the tiling, there are even number of T-tetrominos

$\Rightarrow 8|mn$ (why?) □

7. The unit cube $\mathcal{C} = \{(x, y, z) | \ 0 \leq x, y, z \leq 1 \}$ is cut along the planes $x = y$, $y = z$, and $z = x$. How many pieces are there ?

Solution$_1$ *:*

Cutting $\mathcal{C}$ along $x = y$, we get $Z_1 = \{(x, y, z) \in \mathcal{C} |\ \mathrm{x} > \mathrm{y}\ \}$,
$Z_2 = \{(x, y, z) \in \mathcal{C} |\ \mathrm{y} > \mathrm{x}\ \}$.
Cutting $\mathcal{C}$ along $y = z$, we get $X_1 = \{(x, y, z) \in \mathcal{C} |\ \mathrm{y} > \mathrm{z}\ \}$,
$X_2 = \{(x, y, z) \in \mathcal{C} |\ \mathrm{z} > \mathrm{y}\ \}$.
Cutting $\mathcal{C}$ along $x = z$, we get $Y_1 = \{(x, y, z) \in \mathcal{C} |\ \mathrm{x} > \mathrm{z}\ \}$,
$Y_2 = \{(x, y, z) \in \mathcal{C} |\ \mathrm{z} > \mathrm{x}\ \}$.

Consider $Z_i \cap X_j \cap Y_k \quad (1 \le i, j, k \le 2)$
We have $Z_1 \cap X_1 \cap Y_1 = \{(x, y, z) \in \mathcal{C} |\ \mathrm{x} > \mathrm{y} > \mathrm{z}\ \}$
$Z_1 \cap X_1 \cap Y_2 = \emptyset$
$Z_1 \cap X_2 \cap Y_1 = \{(x, y, z) \in \mathcal{C} |\ \mathrm{x} > \mathrm{z} > \mathrm{y}\ \}$
$Z_1 \cap X_2 \cap Y_2 = \{(x, y, z) \in \mathcal{C} |\ \mathrm{z} > \mathrm{x} > \mathrm{y}\ \}$
$Z_2 \cap X_1 \cap Y_1 = \{(x, y, z) \in \mathcal{C} |\ \mathrm{y} > \mathrm{x} > \mathrm{z}\ \}$
$Z_2 \cap X_1 \cap Y_2 = \{(x, y, z) \in \mathcal{C} |\ \mathrm{y} > \mathrm{z} > \mathrm{x}\ \}$
$Z_2 \cap X_2 \cap Y_1 = \emptyset$
$Z_2 \cap X_2 \cap Y_2 = \{(x, y, z) \in \mathcal{C} |\ \mathrm{z} > \mathrm{y} > \mathrm{x}\ \}$.
$\therefore$ There are 6 pieces.

Solution$_2$ *:*
Each of the planes $x = y,\ y = z,\ z = x$ contains the straight line $x = y = z$.
Three distinct planes which pass a line divide the space into 6 regions. □

8. Show that

$$\sum_{j=0}^{n} (-1)^j\, 2^{n-j} \binom{x}{j} = \sum_{j=0}^{n} \binom{n-x}{j}$$

Proof :
Let

$$f(x) = \sum_{j=0}^{n} (-1)^j\, 2^{n-j} \binom{x}{j}$$

$$g(x) = \sum_{j=0}^{n} \binom{n-x}{j}$$

Then (1) $f(x), g(x)$ are polynomials in x with degree n
(2) $f(x) = g(x)$ for $x = 0, 1, 2, \cdots, n$.
check of (2):
For $k = 0, 1, 2, \cdots, n$,

$$
\begin{aligned}
f(k) &= \sum_{j=0}^{n}(-1)^j\, 2^{n-j}\binom{k}{j} \\
&= \sum_{j=0}^{k}(-1)^j\, 2^{n-j}\binom{k}{j} \\
&= 2^n\sum_{j=0}^{k}(-\frac{1}{2})^j\binom{k}{j} \\
&= 2^n(1-\frac{1}{2})^k \\
&= 2^{n-k}
\end{aligned}
$$

$$
\begin{aligned}
g(k) &= \sum_{j=0}^{n}\binom{n-k}{j} \\
&= \sum_{j=0}^{n-k}\binom{n-k}{j} \\
&= 2^{n-k}
\end{aligned}
$$

Thus $f(k)=g(k)$ for $k=0,1,2,\cdots,n$.

This completes the check of (2)

Now (1),(2) $\Rightarrow f(x)=g(x)$ □

9. (1) Let X be an nonempty set,

$F_1=\{\,(A_1,A_2,A_3):\ A_1,A_2,A_3\subset X\,\}$

$F_2=\{\,(B_1,B_2,\cdots,B_7):\ B_1,B_2,\cdots,B_7\subset X\,\}$

Let $f:F_1\to F_2$ defined by

$$
\begin{aligned}
f(A_1,A_2,A_3)=(&A_1\cap\overline{A_2}\cap\overline{A_3}\ ,\ \overline{A_1}\cap A_2\cap\overline{A_3}\ ,\ \overline{A_1}\cap\overline{A_2}\cap A_3\ , \\
&A_1\cap A_2\cap\overline{A_3}\ ,\ A_1\cap\overline{A_2}\cap A_3\ ,\ \overline{A_1}\cap A_2\cap A_3\ , \\
&A_1\cap A_2\cap A_3)\ , \qquad \textit{where}\ \overline{A_1}=X-A_1.
\end{aligned}
$$

Show that f is a $1-1$ function.

(2) Suppose that X is a set with $|X|=n$.

Let $F=\{\,(A_1,A_2,A_3):\ A_1\cup A_2\cup A_3=X\,\}$

$G=\{\,(A_1,A_2,A_3):\ A_1\cup A_2\cup A_3=X,\ A_1\cap A_2\cap A_3=\emptyset\,\}$

$H=\{\,(A_1,A_2,A_3):\ A_1,A_2,A_3\subset X,\ A_1\cap A_2\cap A_3=\emptyset\,\}$

Find $|F|,|G|$ and $|H|$.

Proof (1) :

Suppose $f(A_1,A_2,A_3)=f(B_1,B_2,B_3)$

Then $A_1\cap\overline{A_2}\cap\overline{A_3}=B_1\cap\overline{B_2}\cap\overline{B_3}$

$A_1\cap A_2\cap\overline{A_3}=B_1\cap B_2\cap\overline{B_3}$

$A_1\cap\overline{A_2}\cap A_3=B_1\cap\overline{B_2}\cap B_3$

$$
\begin{aligned}
& A_1 \cap A_2 \cap A_3 = B_1 \cap B_2 \cap B_3 \\
\Rightarrow \quad & A_1 = (A_1 \cap \overline{A_2} \cap \overline{A_3}) \cup (A_1 \cap A_2 \cap \overline{A_3}) \cup (A_1 \cap \overline{A_2} \cap A_3) \cup (A_1 \cap A_2 \cap A_3) \\
& \quad = (B_1 \cap \overline{B_2} \cap \overline{B_3}) \cup (B_1 \cap B_2 \cap \overline{B_3}) \cup (B_1 \cap \overline{B_2} \cap B_3) \cup (B_1 \cap B_2 \cap B_3) \\
& \quad = B_1
\end{aligned}
$$

Similarly, $A_2 = B_2,\ A_3 = B_3$

Hence $(A_1, A_2, A_3) = (B_1, B_2, B_3)$

$\therefore f$ is $1 - 1$.

Solution (2) :

By (1) , different (A_1, A_2, A_3)'s determine different

$(A_1 \cap \overline{A_2} \cap \overline{A_3}\ ,\ \overline{A_1} \cap A_2 \cap \overline{A_3}\ ,\ \overline{A_1} \cap \overline{A_2} \cap A_3\ ,\ A_1 \cap A_2 \cap \overline{A_3}\ ,$

$A_1 \cap \overline{A_2} \cap A_3\ ,\ \overline{A_1} \cap A_2 \cap A_3\ ,\ A_1 \cap A_2 \cap A_3)$'s

(i) Suppose $A_1 \cup A_2 \cup A_3 = X$.

Every element in X falls into one of seven multually disjoint sets :

$A_1 \cap \overline{A_2} \cap \overline{A_3}\ ,\ \overline{A_1} \cap A_2 \cap \overline{A_3}\ ,\ \overline{A_1} \cap \overline{A_2} \cap A_3\ ,\ A_1 \cap A_2 \cap \overline{A_3}\ ,$

$A_1 \cap \overline{A_2} \cap A_3\ ,\ \overline{A_1} \cap A_2 \cap A_3\ ,\ A_1 \cap A_2 \cap A_3.$

$\therefore$ There are 7^n different (A_1, A_2, A_3) in F

$\Rightarrow |F| = 7^n$

(ii) Suppose $A_1 \cup A_2 \cup A_3 = X,\ A_1 \cap A_2 \cap A_3 = \emptyset$.

Every element in X falls into one of six multually disjoint sets :

$A_1 \cap \overline{A_2} \cap \overline{A_3}\ ,\ \overline{A_1} \cap A_2 \cap \overline{A_3}\ ,\ \overline{A_1} \cap \overline{A_2} \cap A_3\ ,\ A_1 \cap A_2 \cap \overline{A_3}\ ,$

$A_1 \cap \overline{A_2} \cap A_3\ ,\ \overline{A_1} \cap A_2 \cap A_3.$

$\therefore |G| = 6^n$

(iii) Suppose $A_1 \cap A_2 \cap A_3 \subset X,\ A_1 \cap A_2 \cap A_3 = \emptyset$.

Every element in X falls into one of seven multually disjoint sets :

$\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}\ ,\ A_1 \cap \overline{A_2} \cap \overline{A_3}\ ,\ \overline{A_1} \cap A_2 \cap \overline{A_3}\ ,\ \overline{A_1} \cap \overline{A_2} \cap A_3\ ,$

$A_1 \cap A_2 \cap \overline{A_3}\ ,\ A_1 \cap \overline{A_2} \cap A_3\ ,\ \overline{A_1} \cap A_2 \cap A_3.$

$\therefore |H| = 7^n$ □